CIB-W18/26-6-1

INTERNATIONAL COUNCIL FOR BUILDING RESEARCH STUDIES AND DOCUMENTATION

WORKING COMMISSION W18 - TIMBER STRUCTURES

DISCUSSION AND PROPOSAL OF A GENERAL FAILURE CRITERION FOR WOOD

by

T A C M van der Put
Delft University of Technology
The Netherlands

MEETING TWENTY - SIX

ATHENS, GEORGIA

USA

AUGUST 1993

ABSTRACT

Based on a polynomial expansion of the failure surface, a general failure criterion, satisfying equilibrium in all directions, was developed for wood long ago (IUFRO Boras 1982) and shown to apply for clear wood. For wood with (small) defects and (local) grain deviations, equivalent strengths can be defined in the main directions and a comparable equivalent failure criterion applies as was shown by K. Hemmer (PhD thesis 1985). It was shown at the last COST-508 meeting that the second order terms of the polynomial failure criterion represent the critical distorsional energy of initial yield (or failure at initial yield when the test becomes unstable at this point). It will be shown that the third order terms represent special hardening effects (due to micro-crack arrest by strong layers), occurring after initial yield, determining ultimate failure in longitudinal direction in a stable test.

As known, the singularity approach of fracture mechanics predicts for the critical energy release rate: $G_c = G_{Ic} + G_{IIc}$ for collinear crack propagation in grain direction. As p.e. mentioned in the RILEM State of the Art report on fracture mechanics of wood, this is against experimental evidence and in stead, the empirical Wu-equation is used for layered composites. It was shown at the COST 508-meeting at Bordeaux that this wrong prediction is due to this singularity method that, by the critical stress intensities, does not satisfy in all cases the failure criterion and, although this method is generally applied in fracture mechanics of materials, it therefore has to be rejected. It further was shown that the Wu-criterion can be derived from oriented (in grain direction) crack propagation of elliptic micro-cracks and is a necessary condition for the (right form of the) energy principle.

It now will be shown that this Wu- (or Mohr-) criterion is also determining the failure criterion of wood, showing the same oriented micro-cracking to be responsible for failure in general.

Based on this criterion, the existing criteria can be explained as the Hankinson, Norris, and Coulomb criterion. A derivation is given of an exact modified Hankinson criterion and of the general form of the higher order constants and how they can (safely) be determined from uni-axial tests in the main plane.

The exact criterion is as easy to apply as the invalid approximations, now used for the Codes.

INTRODUCTION

Failure criteria, like the Norris-, Hoffmann-, Tsai-Wu- criteria etc., can be seen as forms of a polynomial expansion of the real failure surface. This expansion of the failure surface in stress space into a polynomial, consisting of a linear combination of orthogonal polynomials, provides easily found constants (by the orthogonality property) when the expanded function is known, and the row can be extended, when necessary, without changing the already determined constants of the row. When choosing in advance a limited number of terms of the polynomial, up to some degree, the expansion procedure need not to be performed, because the result is in principle identical to a least square fit of the data to a polynomial of that chosen degree. This choise of the number of terms may depend on the wanted precision of the expansion and the practical use.
Based on this principle of a polynomial expansion of the failure surface, the failure criterion is general, satisfying equilibrium in all directions, and was for wood first developed in [1] and the most important aspects can be found in that publication. The in [1] given explanation of the existing criteria and the approximation of the coupling terms as F_{12}, are verified, p.e. in Madison [2], where it was shown that Cowins approximation [11] does not apply for wood.
A general approach for anisotropic, not orthotropic, behaviour of joints, (as punched out metal plates), and the simplification of the transformations by 2 angles as variables, is given in [3].
A confirmation of the results of [1] by means of coherent measurements (in the radial-longitudinal plane) and the generalization to an equivalent, quasi homogenious. failure criterion for wood with small defects is given in [4], showing, as will be discussed here, a determining influence of crack propagation on the equivalent main strengths. There thus is no reason to maintain the used invalid approximations and to not apply this consistent criterion, also for the Codes, for all cases of combined stresses. Thus far only this criterion gives the possibility of a definition of the off-axis strength of anisotropic materials.

A GENERAL FAILURE CRITERION FOR WOOD

A yield-criterion gives the combinations of stresses whereby flow occurs in an elastic-plastic material like wood in compression. When partial flow (of some component) becomes noticeble, while most of the material remains elastic, initial yield occurs where below the material is regarded to be elastic. For more brittle tensile failures in polymers, there also is an initial yield boundary where above the gradual flow of components at peak stresses and micro-cracking may have a simular effect on stress redistribution as flow. It is discussed in [10] that these flow and failure boundaries may be regarded as equivalent elastic-plastic flow surfaces.
The flow- or failure criterion is a closed surface in the stress space (a more dimensional space with coordinates $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6$).
A cut, (p.e. according to figure 1 through the plane of the co-ordinate axes $y = \sigma_1$ and $x = \sigma_2$), will show a closed curve and such a curve always can be described by a polynomial in x and y like:

$$ax + by + cx^2 + dy^2 + exy + fx^3 + gy^3 + hx^2y + ixy^2 + \cdots\cdots = k \qquad (1)$$

Figure - 1. General form of a failure-ellipsoid and definition of positive stresses.

whereby as much as terms can be accounted for, as is necessary for the wanted precision of the description. The surface will be convex because of the normality principle, (the requirement that "plastic" work done must be positive), and higher order terms, causing local peaks on the surface (and thus causing inflection points) are only possible by local hardening effects depending on the loading path and are outside the flow-criterion. These effects can be treated as given in [1] at: "2.2. Hardenings rules" or by the approach of [10].
It can also be seen that the constants f and g are indeterminate and have to be taken zero because, for $y = 0$, eq.(1) becomes: $ax + cx^2 + fx^3 = k$, or in the real roots x_0, $-x_1$, $-x_2$:

$$(x - x_0)\cdot(x + x_1)\cdot(x + x_2) = 0. \tag{2}$$

Because there are only two points of intersection possible of a closed surface with a line, there are only two roots by the intersecting x-axis p.e. $x = x_0$ and $x = -x_1$ and the part $(x + x_2)$, being never zero within or on the surface and thus is indeterminate, has to be omitted. For a real convex surface f is necessarily zero.
The same applies for g, or: $g = 0$ following from the roots of y when $x = 0$.
The equation can systematically be written as stress-polynomial like:

$$F_i\sigma_i + F_{ij}\sigma_i\sigma_j + F_{ijk}\sigma_i\sigma_j\sigma_k + \cdots = 1 \quad (i,\ j,\ k = 1,\ 2,\ 3,\ 4,\ 5,\ 6) \tag{3}$$

In [1] it is shown that clear wood can be regarded to be orthotropic in the main planes and the principal directions of the strengths are orthogonal (showing the common tensor transformations) and higher order terms normally (outer hardening) can be neglected so that eq.(3) becomes:

$$F_i\sigma_i + F_{ij}\sigma_i\sigma_j = 1 \quad (i,\ j,\ k = 1,\ 2,\ 3,\ 4,\ 5,\ 6) \tag{4}$$

In [10] it is shown that this equation represents the critical distorsional energy of failure. For reasons of energetic receprocity $F_{ij} = F_{ji}$ $(i \neq j)$ and by orthotropic symmetry in the main planes (through the main axes along the grain, tangential and radial) there is no difference in positive and negative shear-strength and terms with uneven powers in σ_6 thus are zero or: $F_{16} = F_{26} = F_6 = 0$; and there is no interaction between normal- and shear-strengths or: $F_{ij} = 0$ $(i \neq j;\ i,\ j = 4,\ 5,\ 6)$.
Thus eq.(4) becomes for a plane stress state in a main plane:

$$F_1\sigma_1 + F_2\sigma_2 + F_{11}\sigma_1^2 + 2F_{12}\sigma_1\sigma_2 + F_{22}\sigma_2^2 + F_{66}\sigma_6^2 = 1 \tag{5}$$

For a thermodynamic allowable criterion (positive finite strain energy) the values F_{ii} must be positive and the failure surface has to be closed and cannot be open-ended and thus the interaction terms are constrained to:

$$F_{11}F_{22} > F_{12}^2. \tag{6}$$

For the uniaxial tensile strength $\sigma_1 = X$ ($\sigma_2 = \sigma_6 = 0$) eq.(5) becomes:

$$F_1\sigma_1 + F_{11}\sigma_1^2 = 1 \quad \text{or:} \quad F_1X + F_{11}X^2 = 1 \tag{7}$$

and for the compression strength $\sigma_1 = -X'$ this is:

$$F_1X' - F_{11}X'^2 = 1 \tag{8}$$

and it follows from eq.(7) and (8) that F_1 en F_{11} are known:

$$F_1 = \frac{1}{X} - \frac{1}{X'} \quad \text{and} \quad F_{11} = \frac{1}{XX'} \tag{9}$$

In the same way is for $\sigma_1 = \sigma_6 = 0$ in the direction perpendicular:

$$F_2 = \frac{1}{Y} - \frac{1}{Y'} \quad \text{and} \quad F_{22} = \frac{1}{YY'} \tag{10}$$

Further is follows for $\sigma_1 = \sigma_2 = 0$ (pure shear), for the shear strength S, that:

$$F_{66} = 1/S^2 \tag{11}$$

and is according to eq.(6): $- 1/\sqrt{XX'YY'} < F_{12} < + 1/\sqrt{XX'YY'}$ (12)

It can be shown (as discussed in [1]) that the restricted values of $2F_{12}$, based on assumed coupling according to the deviator stresses, as given by Norris [13], Hill or Hoffmann [14] as: $2F_{12} = -1/2XY$ or: $2F_{12} = -(1/X^2 + 1/Y^2 - 1/Z^2)$ are not general enough for orthotropic materials and don't apply for wood. There also is no coercive reason to restrict F_{12} according to p.e. Tsai and Hahn [15] as: $2F_{12} = -1/\sqrt{XX'YY'}$, or according to Wu and Stachurski [16] as: $2F_{12} \approx -2/XX'$. These values suggest that $2F_{12}$ is ~ 0.2 to 0.5 times the extreme value of eq.(12). The properties of a real physical surface have to be independent on the orientation of the axes and therefore the tensor transformations apply for the stresses σ of eq.(5). These transformation are derivable from the equilibrium of the stresses on an element formed by the rotated plane and the original planes, or simply, by the circle of Mohr construction. For the uni-axial tensile stress then is:

$$\sigma_1 = \sigma_t\cos^2\vartheta, \quad \sigma_2 = \sigma_t\sin^2\vartheta, \quad \sigma_6 = \sigma_t\sin\vartheta\cos\vartheta.$$

Substitution in eq.(5) gives:

$$F_1\sigma_t\cos^2\vartheta + F_2\sigma_t\sin^2\vartheta + F_{11}\sigma_t^2\cos^4\vartheta + (2F_{12} + F_{66})\sigma_t^2\cos^2\vartheta\sin^2\vartheta + F_{22}\sigma_t^2\sin^4\vartheta = 1 \tag{13}$$

and substitution of the values of F:

$$\sigma_t\cos^2\vartheta\left(\frac{1}{X} - \frac{1}{X'}\right) + \sigma_t\sin^2\vartheta\left(\frac{1}{Y} - \frac{1}{Y'}\right) + \frac{\sigma_t^2\cos^4\vartheta}{XX'} + 2F_{12}\sigma_t^2\sin^2\vartheta\cos^2\vartheta + \frac{\sigma_t^2\sin^4\vartheta}{YY'} + $$
$$+ \frac{\sigma_t\sin^2\vartheta\cos^2\vartheta}{S^2} = 1 \tag{14}$$

It can be seen that for $\vartheta = 0$ this gives the tensile- and compressional strength in p.e. the grain direction: $\sigma_t = X$ en $\sigma_t = -X'$, and for $\vartheta = 90^0$, the tensile and compressional strength perpendicular to the grain: $\sigma_t = Y$ and $\sigma_t = -Y'$, and that a definition is given of the tensile and compressional strength in every direction. These are the points of intersection of the rotated axes with the failure surface. Eq.(13) thus can be read in this strength component along the rotated x-axis: $\sigma_t = \sigma_1$ according to:

$$F_1'\sigma_1 + F_{11}'\sigma_1^2 = 1 \tag{15}$$

The same can be done for the other strengths giving the definition of the transformations of F_i and F_{ij}. The transformation of F_{ij} is also a tensortransformation (of the fourth rank) thus following from the rotation of the symmetry axes of the material. Transformation thus is possible in two manners. The stress-components can be transformed to the symmetry directions according to eq.(5). Or the symmetry axes can be rotated, leaving the stresses along the rotating axes unchanged. For this case the general polynomial expression eq.(16) applies:

$$F'_1\sigma_1 + F'_2\sigma_2 + F'_{11}\sigma_1^2 + 2F'_{12}\sigma_1\sigma_2 + F'_{22}\sigma_2^2 + F'_{16}\sigma_1\sigma_6 + F'_{26}\sigma_2\sigma_6 + F'_{66}\sigma_6^2 = 1 \qquad (16)$$

These transformations of F' are p.e. given in [1].

Transverse strengths

In [1] it is shown that for rotations of the 3-axis, when this axis is chosen along the grain, eq.(5) and (16) may precisely describe the peculiar behaviour of the compression- tension- and (rolling) shear-strength perpendicular to the grain and the off-axis strengths without the need of higher order terms.

When for compression the failure limit is taken to be the stress value after that the same, sufficient high, amount of flow strain has occured, then the differences between radial- tangential- and off-axes strengths may disappear and one directional independent strength value remains (see fig. 2). For tension perpendicular to the grain, only in a rather small region (around 90^0, see fig. 2) in the radial direction the strength is higher and because in practise, the applied direction is not precisely in that direction, there is some freedom, in timber, to choose the weakest plane for failure and the lower bound of the strength will apply being independent of the direction. This means that:

$F_1 - F_2 = 0$ and $F_{11} - F_{22} = 0$

and that also F_{12} is limited according to:

$2F_{12} = F_{11} + F_{22} - F_{66}$.

Further then also is:

$F'_6 = 0$ and $F'_{66} = F_{66} = 1/\tau_{rol}^2$.

From measurement it can be derived that F_{12} is small leading to:

$F_{66} \approx F_{11} + F_{22}$ or τ_{rol} is bounded by:

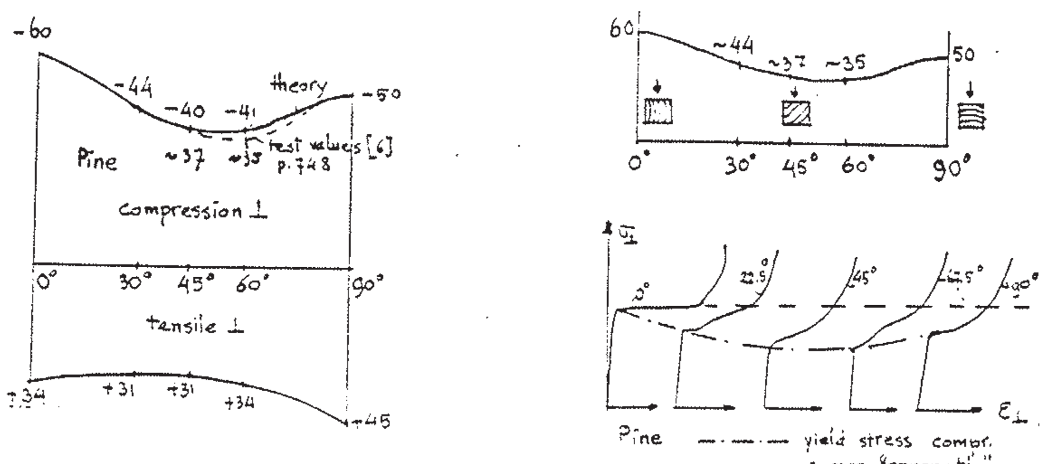


Figure 2 - Yield stresses and hardening

$\tau_{rol} = \sqrt{XX'/2} = \sqrt{YY'/2}$

and the behaviour can be regarded to be quasi isotropic in the transverse direction. The measurements further show for this rotation around the grain-axis that the "shear strengths" in grain direction in the radial- and the tangential plane, F_{44} and F_{55}, are uncoupled or $F_{45} = 0$, as is to be expected from symmetry considerations.

Longitudinal strengths

When now the 3-axis is chosen in the tangential or in the radial direction the same relations apply (with indices 1,2,6) as in the previous case. The equations for this case then give the strengths along and perpendicular to the grain and the shear-strength in the grain direction.

When the test remains stable above initial yield, a type of hardening may occur and third order terms are needed, according to eq.(3), to describe the behaviour. In [1] it was shown (by tests of [11] with σ_2 and σ_6 only), that the longitudinal shear strength in the radial plane increases with compression perpendicular to this plane according to the coupling term F_{266} (direction 2 is the radial direction; direction 1 is in the grain direction):

$$F_2\sigma_2 + F_{22}\sigma_2^2 + F_{66}\sigma_6^2 + 3F_{266}\sigma_2\sigma_6^2 = 1 \quad \text{or:} \quad \frac{\sigma_6}{S} = \sqrt{\frac{(1 - \sigma_2/Y)\cdot(1 + \sigma_2/Y')}{1 + c\sigma_2/Y'}} \qquad (17)$$

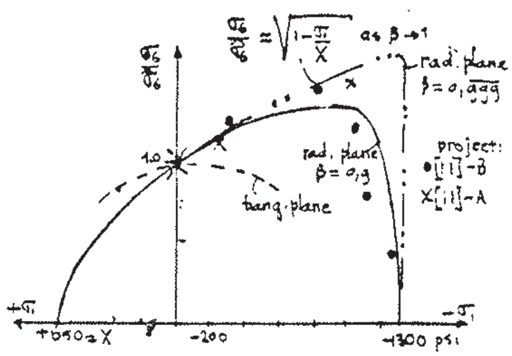
with: $c = 3F_{266}Y'S^2 \approx 0.9$ (0,8 à 0.99).

It is seen from fig. 3, that $c < 1$ is necessary to have a closed surface and thus is determining for the upper bound of F_{266}.

When c approaches $c \approx 1$ (measurements of project A in fig. 3) eq.(17) becomes:

$$\left(\frac{\sigma_6}{S}\right)^2 + \frac{\sigma_2}{Y} \approx 1 \qquad (17')$$

being the Mohr equation or the Wu-equation of fracture mechanics for mixed mode I - II failure (when expressed in stress intensity factors).



$3F_{266} \approx 0.9/S^2Y'$

Figure 3 - Combined shear-tension and shear-compression strengths.

This equation (17') can fully be explained by collinear micro-crack propagation in grain direction [9]. As derived in [9], eq.(17') does not only apply for shear with tension but also for shear with compression σ_2 perpendicular to the flat crack.

For a high stress σ_2 the crack is closed at $\sigma_2 = \sigma_c$ and the crack tip notices only the infuence of $\sigma_2 = \sigma_c$ because for the higher part of σ_2 the load is directly transmitted through the closed crack and eq.(17') becomes:

$$\frac{\sigma_6}{S} = \frac{-\mu(\sigma_2 - \sigma_c)}{S} + \sqrt{1 - \frac{\sigma_c}{Y}} \quad \text{or:} \quad \sigma_6 = C + \mu|\sigma_2| \qquad (17'')$$

where σ_2 and σ_c are negative, giving the Coulomb-equation with an increased shear capacity due to friction: $\mu\sigma_2$. This effect will be small for wood. For micro-cracks the crack closure stress σ_c will have about the value the tensile strength: $\sigma_c \approx - Y$. For maximal compression (at maximal shear), $\sigma_2 \approx - 0.9Y'$, the shear strength will be maximal raised by a factor:$(1 + \mu(0.9Y' - Y)/S)$. Inserting measurements of [4] this factor is maximal: $(\sqrt{2} + 0.3(0.9\cdot 5.6 - 3.7)/9.8)/\sqrt{2} = 1.03$ with respect to the failure without Coulomb friction. Because the possible parabolic fit by equation (17') the co-efficient of friction (above $\sigma_2 = \sigma_c \approx - Y$) is probably smaller than 0.3. It is thus seen that Coulomb friction is neglectable and does not increase the shear-strength by compression perpendicular. This increase is an equivalent hardening effect, caused by crack arrest by the strong layers, so that failure only is possible by longitudinal crack propagation according to the Wu-equation. At higher stresses σ_2, compressional plasticity perpendicular to the grain (project A of [11]), or earlier instability of the test, (project B of [11] with oblique-grain compression tests) may become determining, showing a lower value c of eq.(17) than c = 1.

Because the slopes of the lines (at small σ_2) of project A and B of [11] are the same, there is no indication of an influence of the higher order terms: F_{112}, F_{122} F_{12} and F_{166} of proj. B. Further the line of B is below the line of A and the c-value of B is lower, closer to the eliptic failure criterion. This strongly is an indication that hardening after initial yield (thus departure from the elliptic equation) of project B, the oblique-grain compression test, is much less than that of project A, and thus that the test is less stable. (Project C of [11] follows the elliptic failure criterion because of the influence of transverse failure due to rolling shear that is shown before to be elliptic).

The theory also explains why the parabolic Wu-equation of increased shear strength by transverse compression, did not occur in the tangential plane. If, there are large initial cracks in the tangential plane, (or trans-wall failures), the crack-closure-stress σ_c is small and when also μ is small: σ_6 is about constant in the compressional region (independent on the magnitude of the compression perpendicular) untill there is influence of plasticity by compressional failure. This ultimate shear condition in the weak plane, independent of compression, is predicted by the singularity method and is the basis of the explantion of the Hankinson formula eq.(19), for n = 2, by this method [9]. As shown later n = 2 means that there are no higher order coupling terms (no higher than order 2). Because there thus is no influence of (the equivalent) F_{266} in the tangential plane (F_{355}), F_{266} of the radial plane diminishes quickly at axis-rotation (around the 1-axis), and this higher order effect is only a local effect, only noticeble when loading is in a plane rather close to the radial plane. This thus will hardly be noticeble in timber when failure is free to occur in the weakest plane. The high measured value of F_{266} (measured with $\sigma_1 = 0$) may indicate that for clear wood, F_{122} will be zero in the radial plane according to eq.(22). It also can be deduced from published Hankinson lines of clear wood that F_{122} and F_{266} can be zero in the tangential plane, (confirming the results of proj.

A and B of [11], mentioned above). For these Hankinson lines, $n \approx 2$ in eq.(19), showing all higher order terms to be zero. There is an indication that this is a general property of timber [11], because when shear failure is free to occur the plane of minimum resistance, as usually in large timber beams and glulam, it occurs in the tangential plane and $n = 2$, showing no higher order terms. On the other hand, this needs not to be a general property for all species and tests because it generally is mentioned by Kollmann that $n \approx 2.5$ for compression of clear wood showing that F_{112} and F_{122} will in general not be zero outside the radial plane. This may indicate some hardening by confined dilatation (depending on the test) as discussed in [10]. Because of the oriented crack-propagation, explaining eq.(17), $F_{166} \approx 0$ for clear wood because σ_1 is in the same direction as the flat crack and thus not influenced by that crack.

It was shown in [10] that eq.(5) represents the initial yield condition, being the extended critical distorsional energy principle. Because at initial yield both, the elastic and yield conditions should be satisfied, the second order polynomial applies. For elastic behaviour, this follows from the Hankinson equations (with $n = 2$) that apply for the moduli of elasticity and because the modulus is proportional to the yield strengths, the Hankinson equations (18) apply for initial yield (being the second order polynomial). As mentioned in [10], for glulam in general, and for clear wood in tension, $n \approx 2$ and there are measurements, depending on the type of test, indicating that $n \approx 2$ is possible for compression of clear wood as well (by the "Shereisen" test) or that $n \approx 2$ in the neighbourhood of the tangential plane (following from oblique-grain compression tests), showing therefore no higher order terms and thus failure after initial yield. Thus the second order polynomial then also gives the failure condition. For this case, of no influence of higher order terms, eq.(13) or (14) applies, for the off-grain-axis tensile- and compressional- strengths and eq.(14) can be resolved into factors as follows:

$$\left(\frac{\sigma_t \cos^2\vartheta}{X} + \frac{\sigma_t \sin^2\vartheta}{Y} - 1 \right) \cdot \left(\frac{\sigma_t \cos^2\vartheta}{X'} + \frac{\sigma_t \sin^2\vartheta}{Y'} + 1 \right) = 0 \qquad (18)$$

giving the product of the Hankinson equations for tension and for compression, (where X and X'are the strengths in grain direction). This is possible when:

$2F_{12} + 1/S^2 \approx 1/X'Y + 1/XY'$

In this equation, derived in [1], $(1/X'Y + 1/XY')$ is of the same order and thus about equal to $1/S^2$ and $2F_{12}$ is of lower order with respect to $1/S^2$. In [2] this equation of [1] was used as a measure for F_{12}, thus: $2F_{12} = 1/X'Y + 1/XY' - 1/S^2$ what is a difference of two higher order quantities and thus gives no information of the value of F_{12} that also can be neglected. In [17], wrongly the sum of: $1/S^2$ and $(1/X'Y + 1/XY')$ is taken to be equal to $2F_{12}$, being of higher order with respect to the real value of F_{12} and it is evident that this value did not satisfy eq.(12). Eq.(18) shows that the exponent n of the general Hankinson formula eq.(19):

$$\frac{\sigma_t \cos^n\vartheta}{X} + \frac{\sigma_t \sin^n\vartheta}{Y} = 1 \qquad (19)$$

is $n = 2$ for tension. For compressional failure X and Y have to be replaced by resp. X' en Y' in eq.(19) and eq.(18) shows that $n = 2$ for compression as well when there are no higher order terms.

The equations for timber with defects are in principle derivable from the clear wood equations by analysing the stresses around knots, cracks etc. Descriptive, by the polynomial approach, it is also possible to regard the many possible complicated stress states leading to failure in timber with defects, when loaded in a direction as the strength by the mean stresses in that direction. In that case combined stresses determine the axial strengths. Where due to the grain- and stress deviations, the axial strength is determined by combined shear with tension perpendicular to the grain, stable crack propagation causing an increase of the effective shear-strength (according to the Wu-equation) may occur. Because always shear is involved in failure, also the higher order terms for (apparent pure) normal stresses show this parabolic increase of the effective strength, and the higher order terms are no longer neglectable.
For wood with small defects n can be as low as $n \approx 1.6$ in eq.(19) for tension showing higher order terms to be no longer neglectable for tension. This also has to be expected when knots and defects show a deviation of the grain-direction because the line of the strength has to shift according to that deviation. For compression about the same $n \approx 2.5$ can be expected because the strength depends on the mean grain-direction at yield. In [4] it is shown that F_{166}, F_{266} and F_{112} of the radial plane have influence, what is shown here to represent the equivalent hardening effect due to crack arrest. Eq.(18) thus needs to be extended to account for the smaller Hankinson value of $n < 2$ for tension and $n > 2$ for compression.
An equation of the fourth degree (eq.(21) in σ_t) can always be written as the product of two quadratic equations, eq.(20). For a real failure surface the roots will be real and because the measurements show that one of the quadratic equations is determining for compression and the other for tension and must be valid for zero values of C_t and/or C_d as well, this factorization leads as the only possible solution to be the product of the "Hankinson equations" for tension and compression:

$$\left(\frac{\sigma_t\cos^2\vartheta}{X} + \frac{\sigma_t\sin^2\vartheta}{Y} - 1 + \sigma_t^2\sin^2\vartheta\cos^2\vartheta C_t\right)\cdot\left(\frac{\sigma_t\cos^2\vartheta}{X'} + \frac{\sigma_t\sin^2\vartheta}{Y'} + 1 + \sigma_t^2\sin^2\vartheta\cos^2\vartheta C_d\right) = 0 \qquad (20)$$

In general eq.(20) thus is (as can be seen by performing the multiplication):

$$F_1\sigma_t\cos^2\vartheta + F_2\sigma_t\sin^2\vartheta + F_{11}\sigma_t^2\cos^4\vartheta + (2F_{12} + F_{66})\sigma_t^2\cos^2\vartheta\sin^2\vartheta + F_{22}\sigma_t^2\sin^4\vartheta + 3(F_{112} + F_{166})\sigma_t^3\cos^4\vartheta\sin^2\vartheta + 3(F_{122} + F_{266})\sigma_t^3\sin^4\vartheta\cos^2\vartheta + 12F_{1266}\sigma_t^4\cos^4\vartheta\sin^4\vartheta = 1 \qquad (21)$$

giving the found, general valid, criterion of [4] where it appeared that F_{1122} and other possible higher order terms can be neglected except F_{1266}.
The values C_t and C_d can be found by fitting of the modified "Hankinson equations" (20) for uni-axial off-axis tension and compression giving the constants:

$$2F_{12} = 1/X'Y + 1/XY' - 1/S^2 + C_t - C_d; \quad 3(F_{112} + F_{166}) = C_t/X' + C_d/X;$$
$$3(F_{122} + F_{266}) = C_t/Y' + C_d/Y \quad \text{and} \quad 12F_{1266} = C_tC_d - 12F_{1122} \approx C_tC_d. \qquad (22)$$

Experimentally it is shown that a fit of the Hankinson eq.(19) always is possible. Thus different n values for tension and compression from $n = 2$ means that there are higher order terms and C_t and C_d are not zero as follows from eq.(22).
It was shown in [1] that F_{12} is small and can not be known with a high accuracy. Small errors in the strength values (X, X', Y, Y', S) may change F_{12} by more than

100 % or even change its sign [1] and the value thus is not important. The data of [4] of the principal stresses in longitudinal tension, being close to initial yield, show F_{12} to be zero at initial yield (when $C_t = C_d = 0$, thus F_{12} will be proportional to $C_d - C_t$). The possible estimate in [4], based on the third degree polynomial for all data, shows F_{12} to be negative and to be of lower order with respect to $1/S^2$, showing F_{12} here to be neglectable, and because also $C_d - C_t$ is of lower order, the equivalent shear-strength S follows from:

$$1/S^2 = 1/X'Y + 1/XY' + (1-\alpha)\cdot(C_t - C_d) \approx 1/X'Y + 1/XY' \tag{23}$$

and consequently:

$$2F_{12} = \alpha(C_d - C_t) \quad (\approx 0), \tag{24}$$

where α is a constant found from fitting. Inserting the estimated values of the strengths of [4], based on all data, ($X = 55.5$; $X' = 43.1$; $Y = 3.7$; $Y' = 5.6$) and for $S = 9.4$ to 10.2, then α has values between $\alpha = 0$ to $\alpha = 1$. A lower value of S as was measured for pure shear ($S = 9$), indicates a positive value of F_{12} and it is seen that F_{12} may easily switch between any (small) value (and thus can be neglected).
For a practical criterion, a safe lower bound should be used that ignores the influence of numerous, still higher order terms, because it is not justified to use a complicated equation to account for only small influences. This also applies for F_{1266} wherefore a good estimate (including neglected highest order terms and thus need not to be bounded) will be eq.(22):

$$12F_{1266} = C_t C_d, \quad (\approx 0),$$

but will be shown to be neglectable.
As mentioned before, F_{166} is small or zero for clear wood. However, because of the grain- and stress deviations, F_{166} will not be zero for timber, because crack extension along the grain has components in longitudinal and transverse directions when there is a grain deviation and F_{166} and F_{266} are connected as components depending on the (local) structure. As crack extension component F_{166} will have a simular bound as given by eq.(17) for F_{266} as follows by replacing the index 2 by 1 and Y by X. Thus:

$$3F_{166} \le 0.99/X'S^2.$$

Determining for wood however will not be this bound but the value of F_{166}, following from eq.(22), when F_{112} is known. F_{112} shows to be high by the form of the failure surface and an estimate of the bound of F_{112} has to be made. This form of the failure surface (for the principal stresses, where it is determined by F_{112}), shows a simular cut-off parabola as F_{266}, indicating a common cause with a value of F_{112} close to its upper bound, as found for F_{266}. A general method to determine this bound of F_{112} is given in [1] (for F_{266}). For the purpose here it is sufficient to satisfy eq.(29) of the following approximation.
The upper bound of F_{112} described above applies for $\sigma_6 = 0$. Because for nearly clear wood, the longitudinal crack extension theory predicts F_{166} and F_{122} to be small, the following equation applies:

$$\sigma_1\left(\frac{1}{X} - \frac{1}{X'}\right) + \sigma_2\left(\frac{1}{Y} - \frac{1}{Y'}\right) + \frac{\sigma_1^2}{XX'} + \frac{\sigma_2^2}{YY'} + 3F_{112}\sigma_2\sigma_1^2 = 1 \tag{25}$$

This can be written:

$$\sigma_1\left(X' - X\right) + \sigma_1^2\left(1 + 3F_{112}\sigma_2 XX'\right) = \left(1 - \sigma_2/Y\right)\cdot\left(1 + \sigma_2/Y'\right)\cdot XX' \tag{26}$$

and, neglecting the first term, it can be seen that this equation reduces to a parabola when about: $3F_{112} \approx 1/XX'Y'$ (when the first term is small). The critical value of the bound of F_{112} (to just have a closed surface) will occur at high absolute values of σ_1 and σ_2 and can be expected to occur in the neighbourhood of $\sigma_1 \approx -X'$. For σ_1 approaching: $\sigma_1 \approx -X'$, the first term of eq.(26) is small with respect to the second term and because the compression strength perpendicular to the grain hardly is effected by the longitudinal stress, this maximal value can be inserted, as a good approximation, in this small term giving:

$$\sigma_1^2\left(1 + 3F_{112}\sigma_2XX' + (X' - X)/(-X')\right) = \left(1 - \sigma_2/Y\right)\cdot\left(1 + \sigma_2/Y'\right)\cdot XX'$$

or:

$$\frac{\sigma_1}{X'} = \sqrt{\frac{(1 - \sigma_2/Y)\cdot(1 + \sigma_2/Y')}{1 + c\sigma_2/Y'}} \qquad \text{where } c = 3F_{112}Y'X'^2 \qquad (27)$$

It can be seen that when $c = 1$, the curve reduces to a parabola and the requirement to have a closed curve is $c < 1$. Thus: $3F_{112} < 1/(Y'X'^2)$. The same may apply at the tensile side giving the same equation (27), when X' is replaced by X, or:

$$\frac{\sigma_1}{X} = \sqrt{\frac{(1 - \sigma_2/Y)\cdot(1 + \sigma_2/Y')}{1 + c\sigma_2/Y'}} \qquad \text{where } c = 3F_{112}Y'X^2 \qquad (28)$$

The found parabolas are equivalent to the Wu-equation for shear with tension or compression. Because for wood with defects there always are deviations of the stress or of the grain for the regarded main directions, there always is shear present and when this shear, in the real material planes, is the cause of the failure then according to the maximal stress criterion (eq.(23) of [10]) σ_1/X of eq.(28) should be replaced by σ_6/S of the main plane. By this replacement eq.(28) is identical to eq.(17) and F_{122} is determined by F_{266} of the real material planes.
More general, when F_{12} and F_{122} are not neglectable, the bound: $c < 1$ becomes:

$$c \approx 3F_{112}X'^2Y' - 2F_{12}X'Y' + 3F_{122}Y'^2X' < 1 \qquad \text{for compression} \qquad (29)$$

where, besides $\sigma_1 \approx -X'$, also $\sigma_2 \approx -Y'$ is substituted in the contribution of the small term, as assumed determining point to just have a closed surface.
In the same way is, at the tensile side (replacing $-X'$ by X):

$$c \approx 3F_{112}X^2Y' + 2F_{12}XY' - 3F_{122}Y'^2X < 1 \qquad \text{for tension} \qquad (29')$$

To connect the longitudinal tension region, where F_{112}, F_{12} and F_{122} are about zero, (when this region is separately regarded), to the longitudinal compression region, where F_{112} dominates, it is necessary that for compression:

$$2F_{12} - 3F_{122}Y' << 3F_{112}X' \qquad (30)$$

However for a precise fit still higher order terms (F_{1222}, F_{1112}, F_{1122}) are necessary. With the estimates of F_{266} and F_{112} to be close to their bounds for compression, and with zero normal coupling terms for tension, all constants of the general failure criterion eq.(21) are known, according to eq.(22), depending on C_d and C_t from uni-axial off-axis tension- and compression- tests.
Performing always the stress-transformation to the main planes, as done here, only simple transformation rules (circle of Mohr) have to be known for application.

Estimation of the polynomial constants by uni-axial tests

In fig. 4, a determination of C_d and C_t is given. In this figure of [4], the measurement $Y'/X' = 0.204$ is reduced to obtain a value of $Y'/X' = 0.13$ (at 90^0) to be able to use the measured constants of the bi-axial tests. It is not mentioned how that possibly can be done but the drawn lines in the figure give the prediction of the uni-axial values based on the measured constants according to the general eq.(21) (given in [4] in the strength tensor form as given here by eq.(15)). For comparison the fits of the Hankinson equations are given here, following these drawn lines.
For tension the equivalent Hankinson equation (20) becomes (by scratching the non zero term of the product):

$$\frac{\sigma_t \cos^2\vartheta}{X} + \frac{\sigma_t \sin^2\vartheta}{Y} + \sigma_t^2 \sin^2\vartheta \cos^2\vartheta C_t = 1 \quad (31)$$

and this equation fits the line for tension in fig. 4 when $C_t \approx 11.9/X^2$. The Hankinson equation (19) fits in this case for $n \approx 1.8$ and all 3 equations (21), (31) and (19) give the same result although for the Hankinson equations only the main tension- and compression strength have to be known and the influence of all other quantities are given by: n or C_t.
For compression, the same line as found in [4] was found in [1], (see fig. 11 of [1]), by the second order polynomial with the minimal possible value of F_{12} (according to eq.(12)), showing that except a negative F_{122} (as used in [4]) also a high negative F_{12} may cause the, strong peak at small angles. Because such a peak never is measured, the drawn line of [4] is only followed for the higher angles by the Hankinson equation. For the low angles, the line is drawn through the measured point at $15\,^0$, giving the expectable Hankinson value of $n = 2.4$ of eq.(19) and for eq.(32): $C_d \approx 4/X'^2$. Because of this low measured value, the predicted peak at $10\,^0$

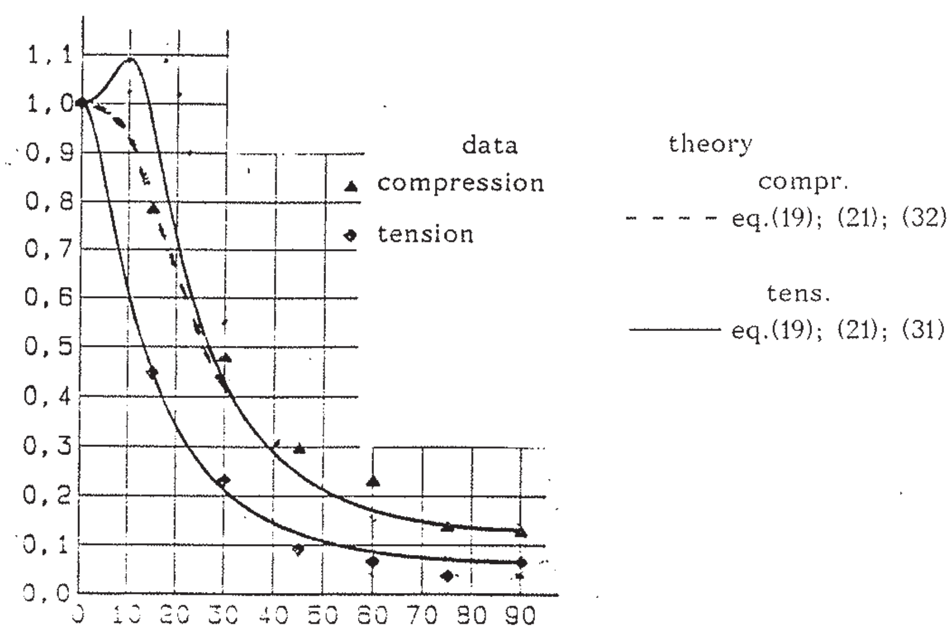


Figure 4 - Uni-axial tension- and compression strengths

is not probable (although theoretically possible, for a high shear strength, to occur at 18^0 in stead of 10^0 with C_d is $7.6/X'^2$ in the Hankinson eq.(32)):

$$\frac{\sigma_t \cos^2\vartheta}{X'} + \frac{\sigma_t \sin^2\vartheta}{Y'} + \sigma_t^2 \sin^2\vartheta \cos^2\vartheta C_d = -1 \qquad (32)$$

This shows that the fit of the polynomial constants, based on the best fit of the measurements of [4], is not well for the oblique grain test. The explanation of this deviation is probably the different state of hardening that can be more or less strong, depending on the stability of the type of test and is less in the Hankinson test. This, for instance, follows from the ratio of the compression strengths perpendicular to the grain and along the grain of 0.2 in the uni-axial tests and 0.1 in the bi-axial tests showing more hardening in the bi-axial test. Further the strong local peak is never measured in the common oblique grain test showing less stability than in the bi-axial test.
An analoguous behaviour occurs in the oblique-grain test of clear wood where the tensile test shows $C_t = 0$ in eq.(20) and the compression test shows C_d to be not zero. A zero value of C_t indicates no higher order terms and thus C_d should be zero. However the tensile test will show unstable fracture at yield what need not to be so for the compression test that may show additional hardening.
Thus the criterion eq.(20) with only $C_t = 0$ may show two different hardening states. For the different hardening states in the two different types of tests, uni-axial and bi-axial, the lowest always possible value should be used for practise.
It thus has to be concluded that the strong hardening in the bi-axial test, will not occur in all circumstances and the hardening parameter F_{112} should be small or omitted for a safe lower bound criterion (according to the oblique grain test).
As generally found in [1] for spruce clear wood, a fit is possible for off-axis tension by a second order polynomial with $F_{12} = 0$. This also applies for wood with defects, as follows from a fit of the data of [4] by the second order polynomial (ellipse) for the principal stresses σ_1 and σ_2 (when $\sigma_6 = 0$), for longitudinal tension ($\sigma_1 > 0$; $F_{12} = 0$), see fig. 5. This means that F_{122} and F_{112} are zero (for $\sigma_1 > 0$) in the radial plane and because the Hankinson value for tension n is different from n = 2, there must be higher order terms for shear (F_{166}, F_{266}).

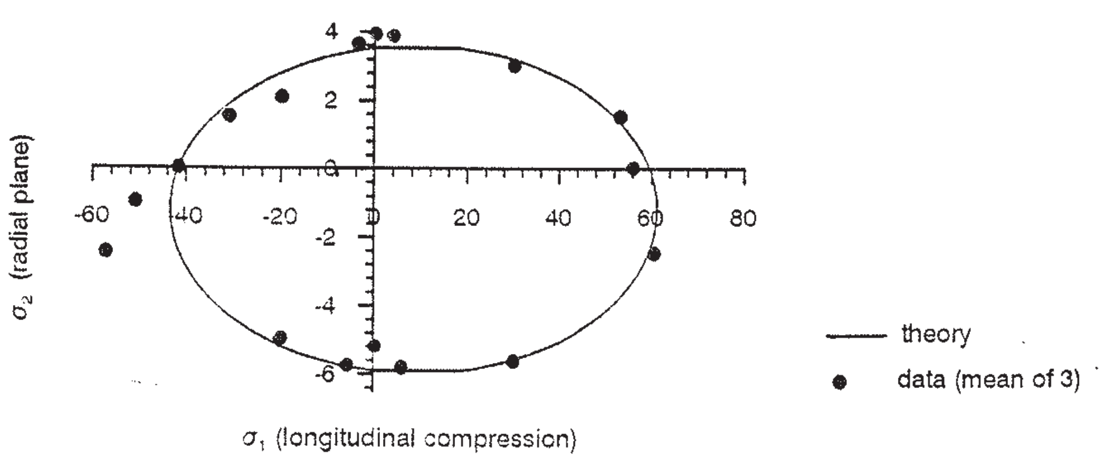


Figure 5 - First yield criterion eq.(5). with $F_{12} = 0$, for $\sigma_6 = 0$.

A first hypothesis thus is (by eq.(22)):
$3F_{266} = C_t/Y'$; $3F_{166} = C_t/X'$, (with $F_{112} = F_{122} = 0$) for tension and (by eq.(22)):
$3F_{122} = C_d/Y$ and $3F_{112} = C_d/X$, (with $F_{166} = F_{266} = 0$) for compression. This gives maximal values of F_{122} and F_{112} for the total fit.
The strength values according to this fit of [4] are (in N/mm^2):
$X = 59.5$; $X' = 46.5$; $Y = 3.5$; $Y' = 5.9$; $S \approx 10$ and with: $C_t = 11.9/X^2$; $C_d = 4/X'^2$; $2F_{12} \approx C_d - C_t$, the predicted values are given in table 1 at column: hyp.1. It is seen that these values fit better than the best fit of the comparable eq.(62) of [4], given in the column indicated with [4]. However for $\sigma_6 = 0$, F_{122} must be negative for a precise fit when $\sigma_1 < 0$ and about zero when $\sigma_1 > 0$, showing that F_{122} has got the function to replace neglected still higher order terms, and a precise fit only can be expected to be possible with multiple higher order terms (with indices 1 and 2). As mentioned in [4], the values of σ_6 can be corrected by F_{1266} to be slightly lower when the sign of σ_1 and σ_2 is the same and to be slightly higher when the sign is opposite. This means that the first and third column value of column: hyp.1 (being 1.1 and 1.0) can be around 1.05. This shows that introduction of F_{1266} only gives a correction of a few percent and justifies the neglection of F_{1266}. The column values further are slightly too high when $\sigma_1 > 0$ and to low for $\sigma_1 < 0$, indicating that F_{166} is not precise. Neglecting the multiple higher order terms, the hypothesis has to be rejected, because F_{122} is too high, distorting the ellipse (at $\sigma_1 > 0$) too much for high negative values of σ_2 and causing the surface to be open at $\sigma_1 < 0$ and high negative σ_2. It thus is probable that F_{122} is much smaller.
Whithout the higher order terms, F_{122} has to satisfy eq.(29) and the highest possible positive value of F_{122} becomes about 0.0001, being about 5 times smaller than according to first hypothesis. The fit now, with this small positive value of F_{122}, is about comparable with the best fit of [4] (that is based on a negative value of F_{122}), but now satisfies eq.(22) and will not show the compression peak in the Hankinson test (fig. 4). The fit is in total not better than a fit with changed constants and is also in total not better than assuming F_{12}, F_{122} and F_{112} to be zero for $\sigma_1 > 0$. This leads to the second hypothesis that the higher order terms for normal stresses are small at fracture (that thus is close to initial yield when $\sigma_1 > 0$) and can be neglected.
In table 1, column hyp.2, the fit is given for $F_{12} = F_{112} = F_{122} = 0$. Because the fit does not change much when data above the uni-axial compression strength: $X' = 41,7$ are neglected, the fit is based on this value and column hyp.2 gives the prediction of failure by the same hardeining state as in to the oblique-grain test (where the strong compressional hardening does not occur). The constants are:

$C_t = 11.9/X^2 = 11.9/59.5^2 = 0.00336$; $C_d = 4/X'^2 = 4/41.7^2 = 0.00230$ and

by eq.(22): $3F_{166} = C_t/X' + C_d/X = 0.00336/41.7 + 0.0023/59.5 = 0.000119$ or:

$c_{166} = 0.000119 \cdot 9.7^2 \cdot 41.7 = 0.47$

$3F_{266} = C_t/Y' + C_d/Y = 0.00332/5.95 + 0.0023/3.5 = 0.00122$ or:

$c_{266} = 0.00122 \cdot 9.7^2 \cdot 5,95 = 0.68$

$F_1 = \frac{1}{X} - \frac{1}{X'} = 1/59.5 - 1/41.7 = -0.0072$; $F_{11} = \frac{1}{XX'} = 1/59.5 \cdot 41.7 = 0.00040$

$F_2 = \frac{1}{Y} - \frac{1}{Y'} = 1/3.5 - 1/5.95 = 0.092$; $F_{22} = \frac{1}{YY'} = 1/3.5 \cdot 5.95 = 0.048$ and:

$F_{66} = \frac{1}{S^2} = 1/9.7^2 = 0.0106;\quad F_{12} = F_{112} = F_{122} = 0.$

The only strong deviation from the last supposition (of having small normal stress coupling terms) thus occurs in the torsion tube test for the principal compressional stresses ($\sigma_6 = 0$; $\sigma_1 < 0$; $\sigma_2 < 0$). The form of the curve is parabolic close to the Wu-equation, showing F_{112} to be high. However the fit is not precise because there appears to be also an other hardening effect, raising the longitudinal compression strength by lower and intermediate values of compression perpendicular σ_2 and F_{122} and F_{12} are also needed to describe this additional hardening effect.
For comparison the strength values of the best fit of all data of [4] are regarded: $X = 55.5$; $X' = 43.1$; $Y = 3.7$; $Y' = 5.6$. The shear strength S of 10 is too high as shown by table 1 and is taken to be 9.4. giving a mean factor of 1.0 in the table for this fit. This leads to the relation of S:
$1/S^2 \approx 1.2(1/X'Y + 1/XY') = 1.2(1/(43.1\cdot 3.7) + 1/(55.5\cdot 5.6)) = 1.2(0.00627 + 0.00322) =$
$= 1.2\cdot 0.00949 = 0.0113$, giving the wanted $S \approx 9.4$.
For the constants now is:

$F_1 = \frac{1}{X} - \frac{1}{X'} = 1/55.5 - 1/43.1 = -0.0052;\quad F_{11} = \frac{1}{XX'} = 1/55.5\cdot 43.1 = 0.00042;$

$F_2 = \frac{1}{Y} - \frac{1}{Y'} = 1/3.7 - 1/5.6 = 0.092;\quad F_{22} = \frac{1}{YY'} = 1/3.7\cdot 5.6 = 0.048.$

Further is:
$C_t = 11.9/X = 11.9/55.5 = 0.00386$ and $C_d = 4/X'^2 = 4/43{,}1^2 = 0.00215.$
F_{12} is the only unknown and gives a reasonable fit with $\alpha \approx 1$ in eq.(24) or:
$2F_{12} \approx C_d - C_t = -0.0017.$
This value satifies eq.(30) and eq.(29) for compression, but not eq.(29') for tension, (showing the surface to be open for tension).
By the strong development of cracks, F_{266} and F_{112} will be high, giving:
$3F_{266} = 0.9/S^2Y' = 0.9/9.4^2 5.6 = 0.00184$
and according to eq.(22):
$3F_{122} = C_t/Y' + C_d/Y - 3F_{266} = 0.00386/5.6 + 0.00215/3.7 - 0.00184 = -0.00057$
$3F_{112} = 0.9/5.6\cdot 43.1^2 = 0.000086$, (and consequently $3F_{166} = 0.000042$ with: .
$2F_{12} \approx -0.0014$) gives the best fit for $\sigma_6 = 0$. However, for combined shear, given in table 1, column 3-compr., the values are comparable with those of column [4]

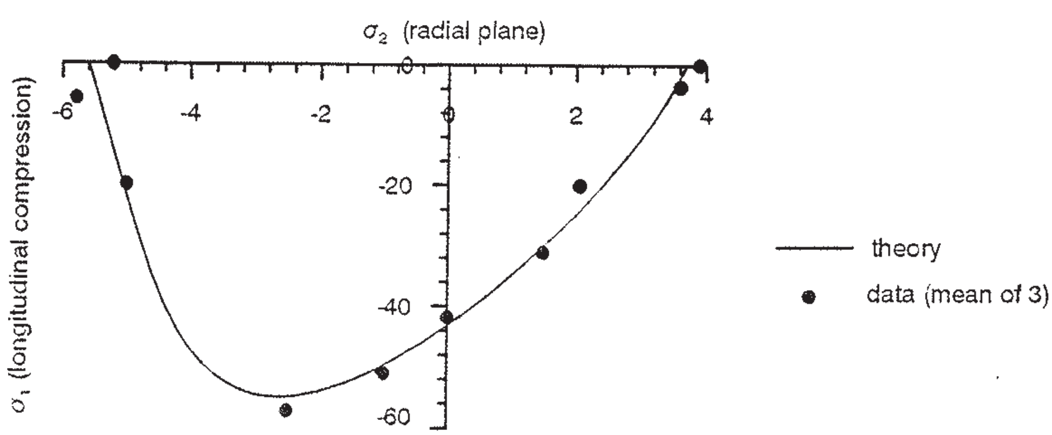


Figure 6 - Yield criterion for compression ($\sigma_1 < 0$) for $\sigma_6 = 0$.

and not well enough for practise. A better fit for the shear strengths is obtained by a slightly reduced factor 0.8 in stead of 0.9 for F_{112} (thus a diminished crack development simular to the oblique-grain test proj. B of fig. 3) giving:
$3F_{112} = 0.8/5.6 \cdot 43.1 = 0.000077$,
$3F_{166} = C_t/X' + C_d/X - 3F_{112} = 0.000128 - 0.000077 = 0.000051$
giving the c-values: $c_{166} = .000051 \cdot 9.4^2 \cdot 43.1 = 0.2$, and: $c_{266} = 0.9$ (starting point).
The combined shear strengths are given in table 1, column 4 (compressional fit), and it is seen that the fit is still less well than according to the 2 foregoing columns due to the negative value of F_{122} as mentioned before.
Further for $\sigma_6 = 0$, the "fit" is given in fig. 6 for compression while for longitudinal tension ($\sigma_1 > 0$) combined with high compression σ_2 perpendicular to the grain, the surface is open and still higher order terms are needed for a closed surface.

Table 1. Shear strength σ_6 for combined normal stresses

σ_1	σ_2	σ_6	factor: $\sigma_{6,theory}/\sigma_{6,test}$				
		test	[4]	hyp.1	hyp.2 tens.	3 compr.	4 compr.
30	1.5	5.8	1.07	1.10	1.03	0.9	1.02
30	0	8.5	0.88	0.97	0.91	0.77	0.92
30	- 2.5	7.9	0.99	1.00	1.10	0.91	1.29
7.3	0	9.2	1.04	1.07	1.03	0.96	1.01
0	2.9	3.7	1.38 !	1.25	1.13	1.39 !	1.19
0	1.5	8.5	0.96	0.95	0.89	0.93	0.86
0	0	9.0	1.11	1.11	1.08	1.04	1.04
0	- 2.5	10.9	0.96	0.98	1.05	0.86	1.07
0	- 5.4	6.8	0.53 !	0.82	1.12	0.45 !	1.12
- 7.7	0	9.6	1.05	1.04	1.01	1.03	0.96
- 20	1.5	7.7	0.84	0.89	0.83	0.93	0.68
- 20	0	9.6	0.99	0.98	0.96	1.10	0.88
- 30	- 2.5	11.3	1.04	0.98	0.90	1.16	0.94
mean factor			.99	1.01	1.0	0.96	1.0

To avoid many higher order terms separate criteria have to be used for longitudinal compression and tension. For compression ($\sigma_1 < 0$) eq.(21) becomes:

$$F_1\sigma_1 + F_2\sigma_2 + F_{11}\sigma_1^2 + F_{22}\sigma_2^2 + 2F_{12}\sigma_1\sigma_2 + F_{66}\sigma_6^2 + 3F_{112}\sigma_1^2\sigma_2 + 3F_{122}\sigma_1\sigma_2^2 + 3F_{166}\sigma_1\sigma_6^2 + 3F_{266}\sigma_2\sigma_6^2 = 1 \qquad (33)$$

Because the C_t, C_d and n-values of the Hankinson equations are sufficiently close to the published extreme values of n, the here calculated c-values can be used in general and inserting F-values in eq.(33), this equation becomes:

$$\frac{\sigma_6^2}{S^2}\cdot\left(1+0.9\cdot\frac{\sigma_2}{Y'}+0.2\cdot\frac{\sigma_1}{X'}\right)=\left(1-\frac{\sigma_1}{X}\right)\cdot\left(1+\frac{\sigma_1}{X'}\right)+\left(1-\frac{\sigma_2}{Y}\right)\cdot\left(1+\frac{\sigma_2}{Y'}\right)+$$

$$-\left(1+0.8\cdot\frac{\sigma_2\cdot\sigma_1^2}{Y'X'^2}-0.77\cdot\frac{\sigma_1\cdot\sigma_2^2}{X'Y'^2}-0.41\cdot\frac{\sigma_1\cdot\sigma_2}{X'Y'}\right) \qquad (34)$$

For tension ($\sigma_1 \geq 0$) eq.(21) becomes (in the radial plane):

$$\frac{\sigma_6^2}{S^2}\cdot\left(1+0.68\cdot\frac{\sigma_2}{Y'}+0.47\cdot\frac{\sigma_1}{X'}\right)=\left(1-\frac{\sigma_1}{X}\right)\cdot\left(1+\frac{\sigma_1}{X'}\right)+\left(1-\frac{\sigma_2}{Y}\right)\cdot\left(1+\frac{\sigma_2}{Y'}\right)-1 \qquad (35)$$

Because the compressional hardening according to eq.(34) only occurs for low values of σ_6, only in the torsional tube test, eq.(35) more generally represents the failure criterion for both tension and compression for the more common loading case as occurs in the oblique-grain test. Neglecting the higher compression strengths far above the uni-axial compression strengths, at $\sigma_6 = 0$, the overall fit is very well and much better than the proposed fit of [4].

For the tangential plane there is a strong indication that the higher order terms are zero (causing n = 2 for timber and glulam). When this is the case, eq.(35) only aplies locally near the radial plane and the mostly determining criterion becomes:

$$\frac{\sigma_6^2}{S^2}-\left(1-\frac{\sigma_1}{X}\right)\cdot\left(1+\frac{\sigma_1}{X'}\right)-\left(1-\frac{\sigma_2}{Y}\right)\cdot\left(1+\frac{\sigma_2}{Y'}\right)=-1$$

or worked out, identical to eq.(5) with $F_{12} = 0$:

$$\frac{\sigma_6^2}{S^2}+\frac{\sigma_1}{X}-\frac{\sigma_1}{X'}+\frac{\sigma_1^2}{XX'}+\frac{\sigma_2}{Y}-\frac{\sigma_2}{Y'}+\frac{\sigma_2^2}{YY'}=1 \qquad (36)$$

It therefore is necessary to use eq.(36) for the Codes in all cases, for timber and clear wood to replace the equivalent, now commonly used, not valid Norris-equations.

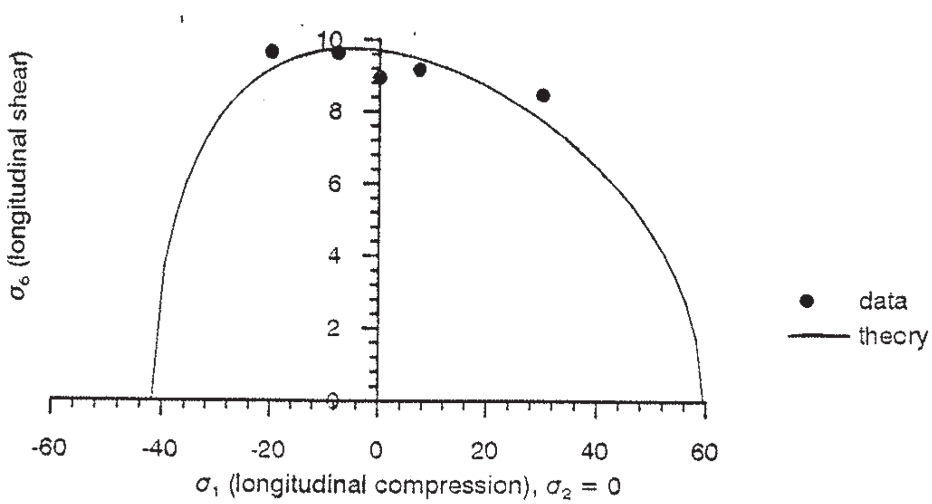


Figure 7 - Combined longitudinal shear with normal stress in grain direction.

For general applications and analysis of test result, the constants can be based on the measured C_d and C_t values or on the Hankinson equations (20) or (19) for the uni-axial stress case when n different from n = 2.
For $\sigma_2 = 0$, determining F_{166}, a plot is given in fig. 7.
For $\sigma_1 = 0$, giving F_{266}, a plot is given in fig. 8.

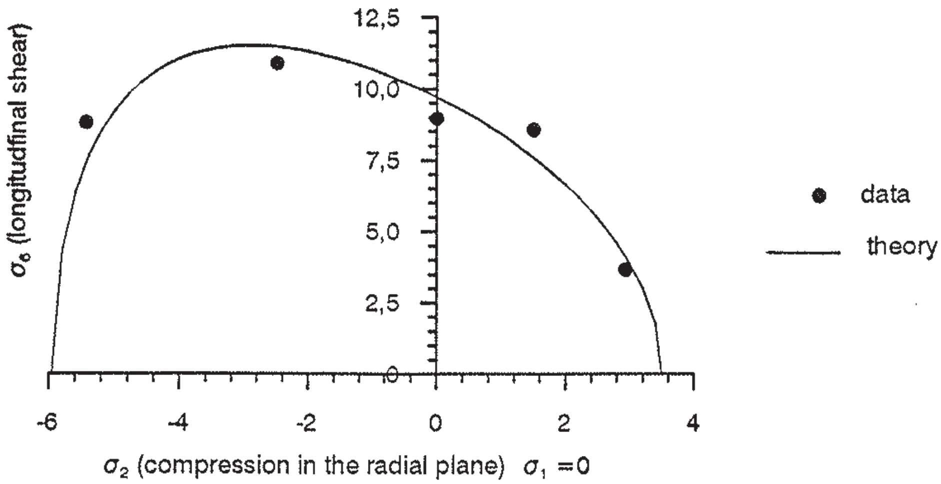


Figure 8 - Longitudinal shear strength ($\sigma_1 = 0$) depending on the normal stress σ_2.

Application for tapered beams

These type of beams are designed according to the theory of orthotropic elasticity and for failure the Norris criterion is used (see eq.(26) of [10]). As shown in [10] this criterion may represent initial yield because, for the determining uni-axial stress state along the boundary, this equation is identical to the Hankinson equation with n = 2. Tests of [5] suggest, for not too small angles α, elastic behaviour up to failure because tension perpendicular is determining. However, for clear wood, for α below 5^0 to 10^0, the elastic values do not apply because of plastic flow in compression (as follows from the much higher value of f_m with respect to f_c) and the used Norris-criterion based on f_m [8] is not right and the theory should be adapted for this case. Further also for higher values of α above about 15^0, the slope may act as a notch and theory should be adapted to account for this, by fracture mechanics determined, strength.
As a scheme, in the elastic range, the beam is regarded as a wedge loaded at the top, wherefore the stress distribution is known [6], [7], what applies at a sufficient distance from the support. Based on this theory, the curve fitting, using one fictive shear strength for longitudinal tension as well as compression, ($f_{c,0} = 44$; $f_{c,90} = 4.0$; $f_v = 5$ MPa. for compression and $f_{t,0} = 44$; $f_{t,90} = 0.6$; $f_v = 5$ MPa, for tension), did not show a good fit. Following [8], by using a value of $f_v = 3.6$ for tension, the fit is good and in accordance with the derived value of the fictive shear-strength of: $f_v = \sqrt{XY/2} = = \sqrt{44 \cdot 0.6/2} = 3.6$. If the given values in [7] and [8] of the uni-axial strengths ($f_{c,0}$, $f_{c,90}$, $f_{t,0}$, $f_{t,90}$) are representative, the best fits for glulam (in the whole range of α), and for clear wood (above $\alpha = 10^0$) are obtained with lower fictive shear strengths of about $f_v = \sqrt{XY/3}$ for tension and higher values of f_v than $f_v = \sqrt{XY/2}$ for compression, showing that higher order terms have an influence and

the Norris-equations (being equivalent to the Hankinson equations for n = 2) have to be replaced by the Hankinson equations (19) with n different from n = 2 (or the exact eq.(20) should be used). This has to be proposed for the future Codes.

CONCLUSION

- All following conclusions apply for the normally used softwoods.
- The tensor polynomial failure criterion can be regarded as an polynomial expansion of the real failure criterion. As a consequence, when a least square fitting procedure is used in stead of the expansion procedure, a, in principle, complete polynomial up to the chosen degree is necessary (whereby terms of this compound polynomial may vanish by general symmetry conditions). Further this fit by a limited number of terms need not to show the precise right values of the expanded components and need not to pass all mean values of the measurements precisely and also p.e. the normality rule and convexity requirement need not to apply exactly.
- In transverse direction a second order polynomial (eq.(36)) is sufficient to describe the strength. When for compression (perpendicular to the grain) the strength is defined as the value after flow and some strain hardening, a lower bound of the overall strength can be chosen that will be directional independent and the behaviour can be regarded to be quasi isotropic in the transverse direction.
- There is a strong indication that for initial yield, or when the tangential plane is determining (as follows from direct measurements and the oblique-grain test), also for the longitudinal strengths a second order polynomial (with F_{12} = 0) is sufficient as yield criterion. When the test becomes unstable early, at initial crack extension, as for instance in the oblique-grain tension test or for compression in the "Shereisen" test (probably also in the radial plane), there are no higher order terms. Higher order terms thus are due to hardening effects (real hardening or equivalent hardening by crack arrest) depending on the type of test that may provide stable or unstable crack propagation after initial yield.
- It is shown that, when the Hankinson parameter n = 2 in eq.(19) for tension and compression, all higher order terms are zero. It is probable that this is a general property for timber [11], because when shear failure is free to occur the plane of minimum resistance, as usually in large timber beams and in glulam, it occurs in the tangential plane, showing no higher order terms.
- For clear wood (and wood with small defects), in a stable test, the longitudinal shear strength in the radial plane increases parabolical with compression perpendicular to this plane depending on the coupling term F_{266} giving the Mohr equation or the Wu- equation, that can be explained by collinear micro-crack propagation in grain direction [9]. This increase is an equivalent hardening effect, due to crack arrest by the strong layers, causing failure only to be possible by longitudinal crack propagation. It is shown that the increase of the shear strength is not due to Coulomb friction, being small for wood.
- Because of the oriented crack-propagation, explaining the Wu-equation, $F_{166} \approx 0$ for clear wood because σ_1 is in the same direction as the flat crack and thus not influenced by that crack. Exept for small clear specimens at compression, (providing a high shear strength by the volume effect), there also is no indication of an influence of the normal coupling terms F_{12}, F_{122} and F_{112} (due to hardening by

confined dilatation).

- For wood with (small) defects and local stress and grain deviations, an equivalent polynomial failure criterion is possible, showing therefore an influence of higher order terms. At least an fourth order polynomial is necessary for a reasonable description. A precise description by a third order polynomial is possible when 2 different criteria are regarded, one for longitudinal tension and for longitudinal compression (simular to the 2 Hankinson equations).
- The general form of the criterion for the uni-axial off-axis strength for wood with defects, is at least determined by a the fourth degree equation (eq.(21) in σ_t) and can always be factorized as a product of two quadratic equations, eq.(20). This leads to extended Hankinson equations, eq.(20), for higher order terms, when n of eq.(19) is different from n = 2.
- Because of grain- and stress deviations, F_{166} will not be zero for timber, as for clear wood, because crack extension along the grain has components in longitudinal and transverse directions when there is a grain deviation and F_{166} and F_{266} are connected as components depending on the (local) structure. As crack extension component F_{166} will show a simular cut-off parabola as F_{266}, indicating the common cause.
- For the same reason, the uni-axial tensile strength in the main direction is determined by the shear strength of the oblique material planes and and F_{112} represents F_{266} of the real material planes, showing the same Wu-parabola. Simular to F_{166}, F_{122} may act as component in transverse direction of F_{112}.
- For wood with defects, when the principal strengths in the main planes (σ_6 = 0) are determining, F_{112}, F_{12} and F_{122} are zero, for longitudinal tension (due to early instability of the test). For combined shear failure (equivalent hardening), there are, for this case, small positve values of F_{112} and F_{122}. It is however shown that for a practical criterion these terms can be neglected and only F_{166} and F_{266} remain for longitudinal tension.

For longitudinal compression at σ_6 = 0, equivalent hardening by crack arrest, (high F_{112}) as well as hardening by confined dilatation (showing a negative F_{122} and F_{12}) may occur. This last type of hardening probably only occurs in the torsion tube test, because the negative F_{122} and F_{12} predict a compression peak (see fig.4) that does not occur in the oblique grain test. For structural elements, this effect thus has to be neglected and the lower bound criterion with only F_{166} and F_{266} (and zero F_{12}, F_{122} and F_{112}) applies also for compression in the radial plane as follows from the good fit.
- Because in the tangential plane, the higher order terms can be zero, the quadratic polynomial eq.(36):

$$\frac{\sigma_6^2}{S^2} + \frac{\sigma_1}{X} - \frac{\sigma_1}{X'} + \frac{\sigma_1^2}{XX'} + \frac{\sigma_2^2}{YY'} + \frac{\sigma_2}{Y} - \frac{\sigma_2}{Y'} = 1$$

should be used as lower bound for the Codes in all cases, for timber and clear wood, and because the equation represents initial yield as well it will apply for the lower 5th percentile of the strength.
- For large sized timber and glulam, where shear failure (or longitudinal tensile failure) may pass radial as well as in tangential directions in the same failure plane, the following (eq.(35)) will apply:

$$\frac{\sigma_6^2}{S^2}\cdot\left(1 + c_{266}\cdot\frac{\sigma_2}{Y'} + c_{166}\cdot\frac{\sigma_1}{X'}\right) = \left(1 - \frac{\sigma_1}{X}\right)\cdot\left(1 + \frac{\sigma_1}{X'}\right) + \left(1 - \frac{\sigma_2}{Y}\right)\cdot\left(1 + \frac{\sigma_2}{Y'}\right) - 1 \qquad (52)$$

where c_{166} and c_{266} follows from oblique-grain tests according to eq.(22) based on the measured C_d and C_t values: $F_{166} = C_t/X' + C_d/X$; $F_{266} = C_t/Y' + C_d/Y$.

- For general applications and analysis of test results, eq.(52) can be used or for the uni-axial loading case, the Hankinson equations (20) or (19).

- The Norris equations are not generally valid and only apply for uni-axial loading, identical to the Hankinson equation with n = 2, when the right (mostly) fictive shear-strength is used. These equations thus should not be used any more.

- Therefore, for tapered beams and for all other cases with determining off-axis uni-axial strength, the general Hankinson equations for tension and compression (with n different from n = 2, depending on the measurements) should be used or the exact equations (31) and (32) (or of course eq.(52)).

REFERENCES

[1] van der Put, T.A.C.M., A general failure criterion for wood, Proc. IUFRO meeting Boras, 1982 (or: Stevinreport 4-82-5 HA-14, Feb. '82).

[2] Liu, J.Y. Evaluation of the tensor polynomial strength of wood, J. of Compos. Mater. Vol. 18, May 1984

[3] van der Put, T.A.C.M., Discussion of the design of nailplate connections, Proceed. Pacific timber eng. conf., Auckland, New Zealand, I.P.E.N.Z. MAY 1984.

[4] Hemmer, K., Versagensarten des Holzes der Weisstanne unter mehrassige Beanspruchung, Dissertation, Karlsruhe 1985.

[5] Möhler K. and Hemmer K., IUFRO timber engineering meeting, augustus 1978.

[6] Lekhnitskii, S.G., Anisotropic plates, Gordon and Break Science Publ., 1968.

[7] Ribberholt H., Tapered timber beams, CIB-W18/11-10-1, meeting Vienna, 1979.

[8] Möhler K., Consideration of combined stresses for lumber and glued laminated timber CIB-W18/9-6-4, 1978 en CIB-W18/11-6-3, 1979.

[9] van der Put, T.A.C.M., Explanation of the mixed mode interaction equation, Proceed. of the Cost 508 workshop on fracture mechanics, Bordeaux 1992.

[10] van der Put, T.A.C.M., Post yielding behaviour of wood, Proc. of the Cost 508 workshop on yield and failure of wood, Limerick 1993.

[11] Keenan, F.J. and Jaeger, T.A., Effect of transverse stress on shear strength and failure mechanism of Douglas-fir, Proc. first int. conf. on Wood fracture, Banff, Alberta, Canada, 1978.

[12] Cowin, S.C., ODn the strength anisotropy of bone and wood, J. Applied Mechanics, ASME Transactions, vol. 46, no 4, 1979.

[13] Norris, C.B., The elastic theory of wood failure, ASME Trans., vol.61, 3, 1939.

[14] Hoffmann, O., J. Composite Materials, p. 200, 1967.

[15] Tsai S.W., and Hahn H.T., Introduction to composite materials, Westport, CT: Technometric Publ. Co., Inc., 1980.

[16] Wu R.Y., Stachurski Z., Evaluation of the normal stress interaction parameter in the polynomial strength theory for anisotropic mater. J. Comp. Mater. 1984.

[17] Leichti R.J., Tang R.C., Predictin the load capacity of wood composite I-beams using the tensor polynomial strength theory. Wood Sci. and Technol. 23, 1989.